Lecture 4

- Recap
- Greens functions for wave operator (cont.)
- Electromagnetic energy
- Electromagnetic waves (plane waves)

Recap

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = \dfrac{\rho}{\varepsilon_0} \\ \vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \mu_0\varepsilon_0 \dfrac{\partial \vec{E}}{\partial t} \\ \vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \\ \vec{\nabla}\cdot\vec{B} = 0 \end{cases}$$

$$\vec{E} = -\vec{\nabla}\varphi - \partial_t\vec{A}$$
$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$\begin{cases} \dfrac{1}{c^2}\partial_t^2\vec{A} - \Delta\vec{A} = \mu_0\vec{J} \\ \dfrac{1}{c^2}\partial_t^2\varphi - \Delta\varphi = \dfrac{\rho}{\varepsilon_0} \\ \dfrac{1}{c^2}\partial_t\varphi + \vec{\nabla}\cdot\vec{A} = 0 \end{cases}$$

$\to$ Gauge condition

Last time we studied Electrostatics (Magneto-)

→ Method of images

$$G_{N/D}(x,x') = \frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{x}'|} \pm \frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{x}'_R|}$$

→ Eigenfunctions:

$$G_{N/D}(x,x') = \sum_n \frac{1}{\lambda_n}\psi_n^{N/D\,*}(x')\,\psi_n^{N/D}(x)$$

Dynamics

$$\varphi(x,t) = \frac{1}{\varepsilon_0}\int d^3x' dt' G_R(\vec{x}, t, \vec{x}', t') \cdot \rho(x', t')$$

$\hookrightarrow$ retarded G.F.

$$G_R \sim \Theta(t-t')$$

To compute the G.F. we will again use Fourier transform

$$\Box G = \delta^3(x-x')\,\delta(t-t')$$

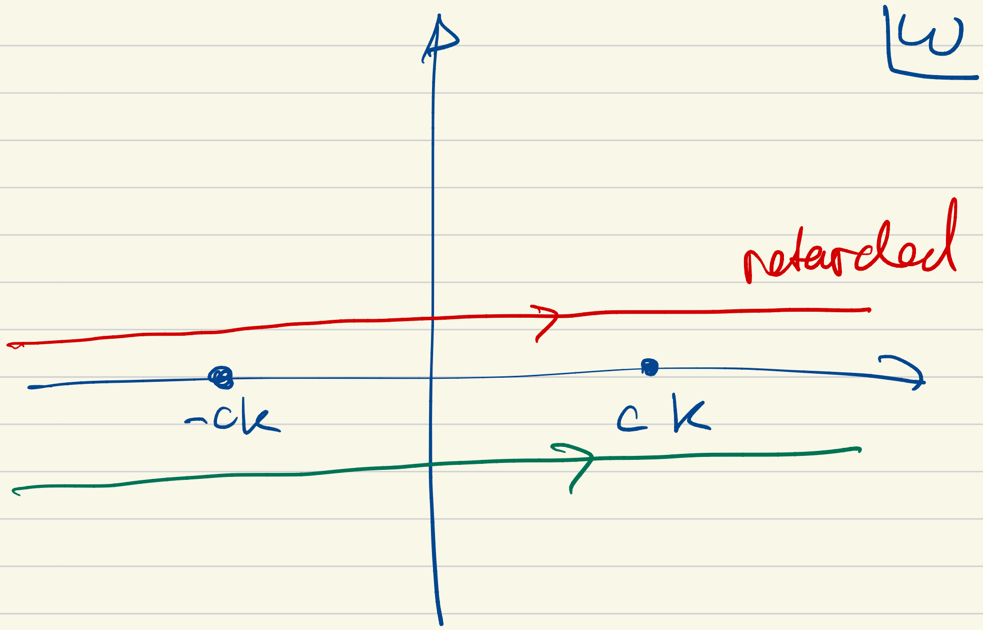
$$\Downarrow$$

$$\left[-\frac{\omega^2}{c^2} + k^2\right]\hat{G}(\omega, k) = e^{-ikx' + i\omega t}$$

$$\hat{G} = \frac{1}{-\frac{\omega^2}{c^2} + k^2}$$

$$G = \int \frac{d^3k\, d\omega}{(2\pi)^4} e^{ikx - i\omega t}\,\hat{G}$$

Let us first take the integral over ω. This integral is not well-defined because of the singularities at $\omega = \pm ck$. These are resolved by changing the contour of integration:

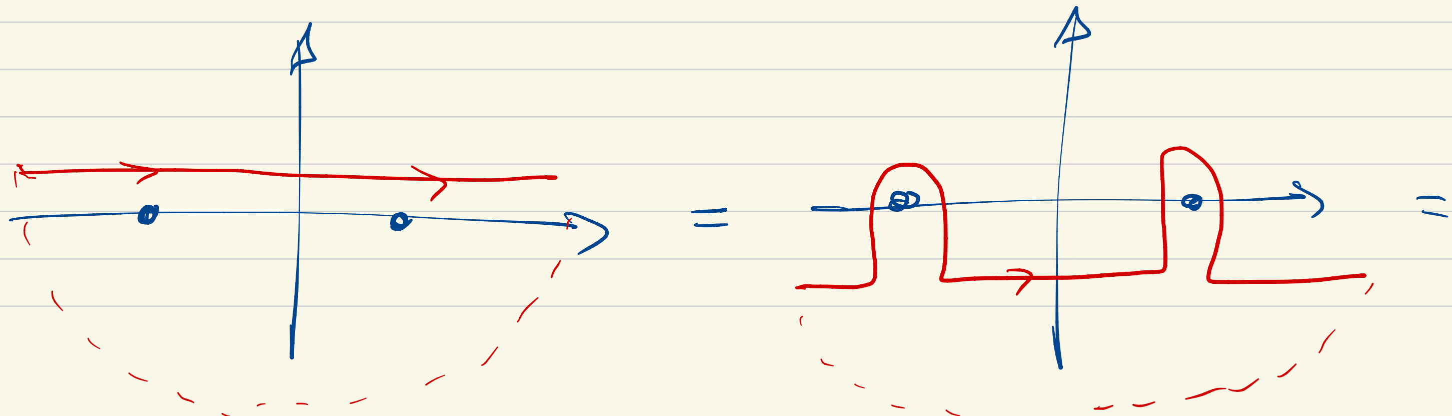


$t < 0$

$i\omega \to -\infty$

$e^{-i\omega t} \to 0$

Now let us consider $t > 0$. Our goal is to deform the contour to simplify the integral.



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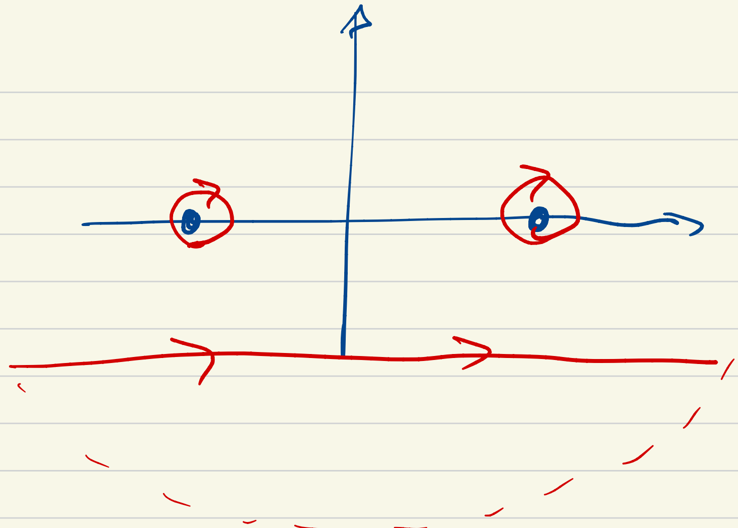

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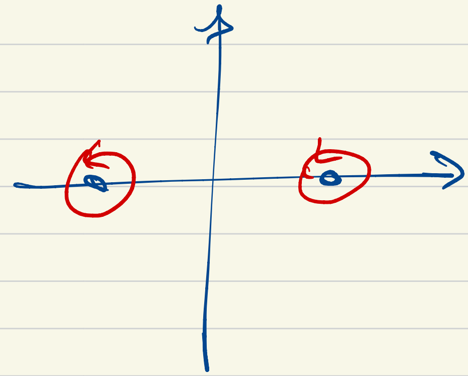


$$\int \frac{d^3k\, d\omega}{(2\pi)^4}\, e^{ik(x-x')-i\omega(t-t')} \frac{c^2}{-(\omega-ck)(\omega+ck)} =$$

$\vec r = \vec x - \vec x'$ $\qquad \omega = ck \quad - \qquad \omega = -ck \quad +$

$$= -2\pi i \cdot \frac{c^2}{(2\pi)^4} \int d^3k\, \frac{e^{i\vec k(\vec x - \vec x')}}{2ck} \left(e^{ick(t-t')} - e^{-ick(t-t')} \right)$$

$$2\pi \int_0^\pi d\theta \sin\theta\, e^{ikr\cos\theta} = \frac{2\pi}{ikr}\left(e^{ikr} - e^{-ikr}\right)$$

$$-\frac{2\pi^3 \cdot c}{r} \cdot \int_0^\infty \frac{dk}{(2\pi)^4} \Big[e^{ik(r+c(t-t'))} +$$

$$+ e^{-ik(r+c(t-t'))} - e^{ik(r-c(t-t'))} - e^{-ik(r-c(t-t'))} \Big]$$

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik\alpha} = \delta(\alpha).$$

$t > t' \Rightarrow$ only one term survives

$$G_R = \frac{\delta(t-t'-\frac{r}{c})}{4\pi r} \qquad G_A = \frac{\delta(t-t'+\frac{r}{c})}{4\pi r}$$

$$\varphi(x) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(x', t - \frac{|x-x'|}{c})}{|x-x'|} d^3x'$$

$$\vec{A}(x) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(x', t - \frac{|x-x'|}{c})}{|x-x'|} d^3x'$$

Let us check explicitly that the GF satisfies the original equation. It is convenient to use spherical coordinates:

$$\left[-\frac{\partial_t^2}{c^2}+\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\right]\frac{\delta\left(t-t'-\frac{r}{c}\right)}{r}=$$

$$=\delta(t-t')\cdot\delta^3(\vec{r})$$

Electrostatic Energy

- Potential energy of a set of charges:

$$U=\sum_{\langle ij\rangle}\frac{q_i q_j}{4\pi\varepsilon_0|r_i-r_j|}$$

In the continuum this becomes

$$U = \frac{1}{2}\int d^3x\, d^3y \,\frac{\rho(x)\rho(y)}{4\pi\varepsilon_0 |x-y|} =$$

$$= \frac{1}{2}\int d^3x\, \rho(x)\, \varphi(x)$$

$$\Delta\varphi(x) = -\frac{\rho}{\varepsilon_0} \Rightarrow$$

$$U = -\frac{\varepsilon_0}{2}\int d^3x\, \Delta\varphi\cdot\varphi =$$

$$= \frac{\varepsilon_0}{2}\int d^3x\, \vec{\nabla}\cdot\varphi\cdot\vec{\nabla}\cdot\varphi = \frac{\varepsilon_0}{2}\int d^3x\, |\vec{E}|^2$$

Note that now a single charge gives some energy

Energy density: $u = \frac{\varepsilon_0 |\vec{E}|^2}{2}$

Electromagnetic energy

Conservation of energy and Maxwell eq. implies the following

$$\vec\nabla\times\vec B = \mu_0 \vec J + \mu_0\varepsilon_0 \frac{\partial \vec E}{\partial t} \qquad \times\vec E$$

$$\vec\nabla\times\vec E = -\frac{\partial \vec B}{\partial t} \qquad \times\vec B$$

$$\mu_0 \vec J\cdot\vec E + \mu_0\varepsilon_0 \frac{1}{2}\frac{\partial \vec E^2}{\partial t} + \frac{1}{2}\frac{\partial \vec B^2}{\partial t} =$$

$$= \underbrace{\vec E\cdot\vec\nabla\times\vec B - \vec B\cdot\vec\nabla\times\vec E}_{-\vec\nabla\cdot(\vec E\times\vec B)} \qquad \Big/\frac{1}{\mu_0}$$

Expanded:

$$\varepsilon^{ijk} E_j\, \partial_i B_k + \varepsilon^{ijk}(\partial_i E_j)\cdot B_k$$

$$S = \frac{1}{\mu_0}\vec E\times\vec B$$

$$\overbrace{\int_V d^3x\, \vec J\cdot\vec E}^{\text{work done on currents}} + \underbrace{\frac{\partial}{\partial t}\int_V d^3x\, u}_{\text{change in energy}} + \overbrace{\int_{\partial V} d\vec\sigma\cdot\vec S}^{\text{energy flux}} = 0$$

Now $u = \frac{\varepsilon_0}{2} |\vec{E}|^2 + \frac{1}{2\mu_0} |\vec{B}|^2$

This can be considered a derivation of energy density when magnetic field is also present.

Plane waves

Now we will work again with the potentials. We will look for the solutions of the homogeneous equation:

$$\Box \varphi = 0$$

$$\Box \vec{A} = 0$$

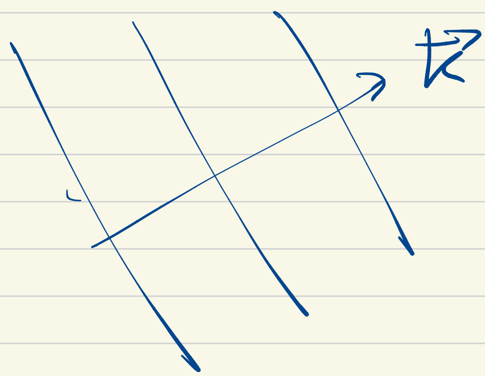
We will look for solutions in the following form:

$$\hat{\varphi}(x,t) = \hat{\varphi}(\vec{k}) e^{-i\varphi} + c.c.$$

$$\hat{A}(x,t) = \hat{A}(\vec{k}) e^{-i\varphi} + c.c.$$

$$\varphi = \omega_k t - \vec{k}\cdot\vec{x}$$

This corresponds to having an EM wave with a special profile that is a single Fourier component.



Plugging the Ansatz gives the equation:

$$\omega_k^2 - c^2 \vec{k}^2$$

$$\omega_k = \pm c|\vec{k}| \quad \text{(we choose + sign)}$$

$$v_{phase} = c$$

Now let us impose the Lorentz gauge condition:

$$\vec{\partial}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0$$

$$\hat{\varphi} = \frac{c^2}{\omega_k}\vec{k}\cdot\hat{\vec{A}} = \frac{c\vec{k}\cdot\vec{A}}{|k|}$$

Now let us find $\vec{E}$ and $\vec{B}$

We use a similar Ansatz:

$\varphi = i\omega t - ikx$

$$\vec{E} = \hat{\vec{E}}(k)\,e^{-i\varphi} + c.c.$$

$$\vec{B} = \hat{\vec{B}}(k)\,e^{-i\varphi} + c.c.$$

$$\vec{E} = -\vec{\partial}\varphi - \dot{\vec{A}} = \left(i\vec{k}\hat{\varphi} - i\omega_k\hat{A}\right)e^{-i\varphi} + c.c.$$

$$\Rightarrow \hat{E} = -ic\frac{\vec{k}}{|k|}\vec{k}\cdot\vec{A} + ic|k|\vec{A} \equiv ick\hat{A}_\perp$$

$$\vec{B} = \vec{\partial}\times\vec{A} \Rightarrow \hat{\vec{B}}_\alpha = i\varepsilon_{\alpha\beta\gamma}k_\beta(\hat{A}_\perp)_\gamma$$

From this we see that three vectors

$\vec{k}$, $\vec{E}$, $\vec{B}$ are all orthogonal :

$$\vec{E}\cdot\vec{k} = \vec{B}\cdot\vec{k} = \vec{E}\cdot\vec{B} = 0$$

Also $\quad B = \frac{1}{c}\vec{n}\times\vec{E}$, $\quad n = \frac{\vec{k}}{k}$